Difference Equations

Difference equations are the discrete equivalent of differential equations. suppose that if there are $n+1$ items of data, the number of instruction cycles wed increase by $10n+1$. Then

$$i[n+1] = i[n] + 10n + 1$$

This is an example of a difference equation. The dependent variable is i; the independent variable is n.

putting $n=0$ in the difference equation gives

$$i[1] = i[0] + 10(0) + 1 = 1$$

Similarly, $i[2]=12$, $i[3]=33$, $i[4]=64$ and so on

There are strong similarities between difference and differential equations. The important point to note is that with difference equations, the independent variable is discrete, not continuous. In the above example, n is the number of pixels; it can have only integer values.

Dependent and independent variables

Consider a simple difference equation

$$x[n+1] - x[n] = 10$$

The dependent variable is x; the independent variable is n. In the difference equation

$$y[k+1] - y[k] = 3k+5$$

the dependent variable is y and the independent variable is k.

Ex: Show $x[n] = A2^n$, where A is a constant, is a solution of $x[n+1] - 2x[n] = 0$

calculate the constant A at $x[0] = 3$

Solution:

$$x[n] = A2^n$$

$$x[n+1] = 2x[n] = 2A2^n$$

Hence

$$x[n+1] - 2x[n] = \cancel{2A2^n} - \cancel{2A2^n} = 0$$

$$x[n] = A2^n \Longrightarrow x[0] = A2^0 = 3$$

$$\therefore A = 3$$

$$x[n] = 3 \cdot 2^n$$

Linear and non-linear equation

An equation is linear if the dependent variable occurs only to the first power. If an equation is not linear it is non-linear. For example,

$$3x[n+1] - x[n] = 10$$

$$y[n+1] - 2y[n-1] = n^2$$

$$K\, z[k+2] + z[k] = z[k-1]$$

All above equations are linear equations. Note that the presence of the term n^2 does not make the equation non-linear, since n is the independent variable. However

$$(x[n+1])^2 - x[n] = 10$$

$$y[k+1] = \sqrt{y[k]+1}$$

are both non-linear. Also

$$z[n+1] \cdot z[n] = n^2 + 100$$

$$\sin x[n] = x[n-1]$$

are non-linear. The product term $z[n+1] \cdot z[n]$ and the term $\sin x[n]$ are the causes of the non-linearity.

Order

The order is the difference between the highest and lowest arguments of the dependent variable. The equation

$$3x[n+2] - x[n+1] - 7x[n] = n$$

is second order because the difference between $n+2$ and n is two. The equation

$$x[n+1] \cdot x[n-1] = 7x[n-2]$$

is third order because the difference between $n+1$ and $n-2$ is three.

The difference equations of order one or greater are referred to as recursive difference equations because their solution requires knowledge of previous values of the dependent variable. The difference equation

$$x[n] = n^2 + n + 1$$

has zero order and so is a non-recursive difference equation.

Homogeneous and inhomogeneous equations

The meaning of homogeneous and inhomogeneous as applied to linear difference equations are analogous to those meanings when applied to differential equations. To decide whether a linear equation is homogeneous or inhomogeneous it is written in standard form, with all the dependent variable terms on the Left hand side (LHS). Any remaining independent variable terms are written on the Right Hand Side [RHS]. For example,

$$3nx[n+1] - 2n^3 = x[n-1] \longrightarrow (1)$$

is written as

$$3nx[n+1] - x[n-1] = 2n^3$$

If the [RHS] is zero, the equation is homogeneous; otherwise it is inhomogeneous. Equation (1) is inhomogeneous but

$$3x[n+1].n - x[n-1] = 0$$

is homogeneous

Ex: For the following equations

i) $2x[n] - 3nx[n-1] + x[n-2] + n^2 = 0$

ii) $\frac{1}{3}(x[n+1] - x[n-1]) = x[n]$

iii) $z[n+2] \cdot (2n - z[n-1]) = n+1$

iv) $\dfrac{7x[n-1]}{x[n-2]} = \dfrac{n+1}{n-1}$

v) $w[n+3] \cdot w[n+1] = n^3 - 1$

vi) $y(n+2) + 2y[n+1] = 6s[n+2] - 2s[n+1] + s[n]$
where y is the dependent variable

vii) $x[k+3] - 2x[k+2] + x[k] = e[k+2] - e[k]$
where x is the dependent variable.

a - state the order of each of the following equs.
b - state whether each equation is linear or non-liner
c - For each linear equation, state whether it is homogeneous or inhomogeneous

Solution

i) second order, linear, inhomogeneous

ii) second order, linear, homogeneous

iii) third order, nonlinear

iv) First order, linear, homogenous

v) Second order nonlinear,

vi) First order, linear, inhomogeneous

vii) Third order, linear, inhomogeneous

Rewriting difference equations

Sometimes an equation or expression can be written in difference ways. At first sight, it may appear there are two independent equations when in fact there is only one. Thus we need to be able to rewrite equation so that comparisons can be made. When general solutions of equations are to be found, usually the equation is first written in a standard form. So once again there is a need to rewrite equations.

Ex: Rewrite the equation so that the highest argumen of the dependent variable is $n+1$.

$$x[n+3] - x[n+2] = 2n \qquad x[2] = 7$$

Solution: The highest argument in the given equation is $n+3$; this must be reduced by 2 to $n+1$. To do this n is replaced by $n-2$. The equation becomes

$$x[(n-2)+3] - x[(n-2)+2] = 2(n-2)$$

$$x[n+1] - x[n] = 2(n-2] \qquad x[2]=7$$

Note, however, that the initial condition, $x[2]=7$, is not changed. This is simply stating that x has a value of 7 when the independent variable has a value of 2.

EX: write the following equations so that the highest argument of the dependent variable is $n+2$.

a- $3x[n+4] - 2n\,x[n+2] = (n-1)^2 \quad x[3]=6,\ x[4]=-7$

b- $z[n-2] + z[n-1] + z[n] = 1+n \quad z[0]=1,\ z[1]=0$

Solution:

a- The highest argument, $n+4$, must be replaced by $n-2$ throughout the equation

$3x[n+2] - 2(n-2)\,x[n] = (n-3)^2 \quad x[3]=6,\ x[4]=-7$

b- The highest argument, n, is increased to $n+2$, that is n is replaced by $n+2$.

$z[n] + z[n+1] + z[n+2] = n+3 \quad z[0]=1,\ z[1]=0$

H.W: write the following equations so that the highest argument of the dependent variable is k.

a- $$a[k+2] = \frac{s[k+2] - 25[k+1] + s[k]}{15}$$

where a is the dependent variable.

b- $$a[k+3] = \frac{w[k+3] + w[k+2] + w[k+1] + w[k] + w[k-1]}{5}$$

where a is the dependent variable.

Block diagram representation of difference equations

Before discussing block diagrams it is necessary to review the topic of sampling. Difference equations operate on discrete-time data and therefore a continuous signal needs to be sampled before use. In the most common form of sampling, a sample is taken at regular intervals, T. A continuous signal and the sequence produced by sampling it are shown in figure (1). Some authors write the sequence as $x(nT)$ to indicate that the sequence has been obtained by sampling a continuous waveform at intervals T. We will not use this convention but simply refer to the sampled sequence as $x[n]$.

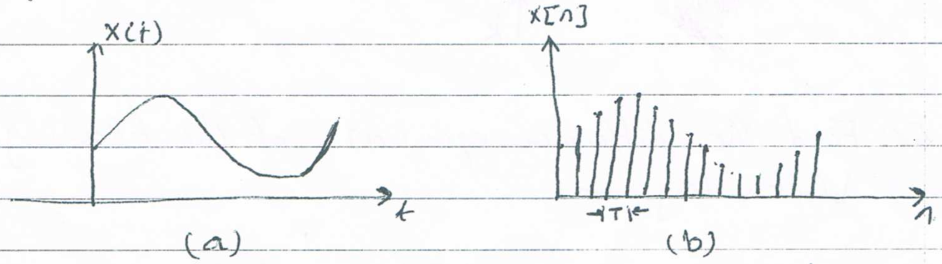


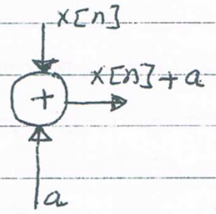
Fig. (1) a- continuous signal b- discrete signal

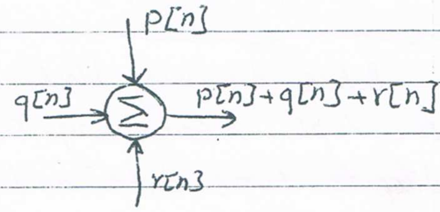
Several components are used in a block diagram.

1- The delay block

T: time interval

2- Summer block

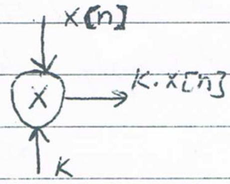




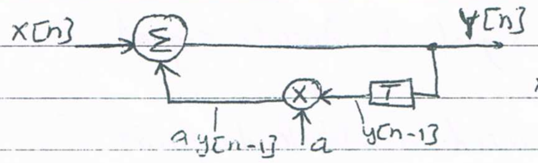
Adding a constant to a sequence

Adding sequences together using a summer

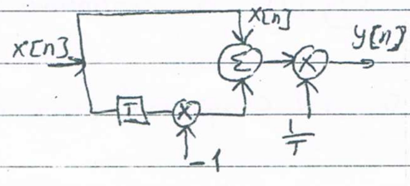
3- Scaling block



Ex: Find the difference equations of the following blocks diagram



(a)



(b)

Solution

a- $y[n] = x[n] + a\,y[n-1] \Rightarrow y[n] - a\,y[n-1] = x[n]$

b- $\{x[n] - x[n-1]\} \cdot \frac{1}{T} = y[n]$

Types of input

1- Delta sequence $\delta(k)$

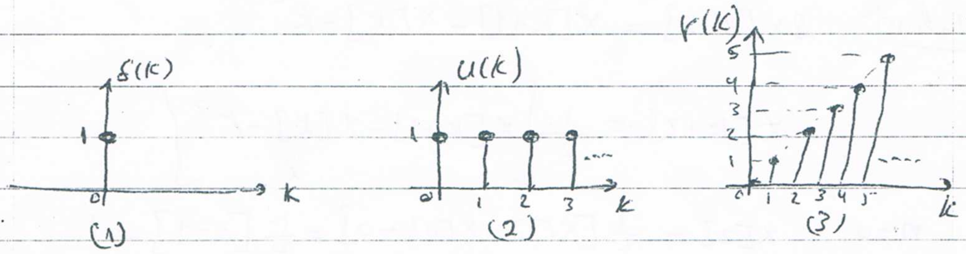
$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

2. Unit step sequence

$$U(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

3- Unit ramp sequence

$$r(k) = k \quad k \in N.$$



1- $\delta(k)$
2- $u(k)$
3- $r(k)$

Numerical solution of difference equations

Having seen how difference equations are formulated we now proceed to methods of solution. The numerical method illustrated may be applied to all classes of difference equation.

EX: Given

$$x[n+1] - x[n] = n \qquad x[0] = 1$$

determine $x[1]$, $x[2]$, and $x[3]$

Solution $\quad x[n+1] = x[n] + n$

$n=0 \quad x[1] = x[0] + 0 = 1 + 0 = 1$

$n=1 \quad x[2] = x[1] + 1 = 1 + 1 = 2$

$n=2 \quad x[3] = x[2] + 2 = 2 + 2 = 4$

EX: Determine $x[4]$ given

$$2x[k+2] - x[k+1] + x[k] = -k^2 \qquad x[0]=1, x[1]=3$$

Solution $\quad 2x[k+2] = x[k+1] - x[k] - k^2$

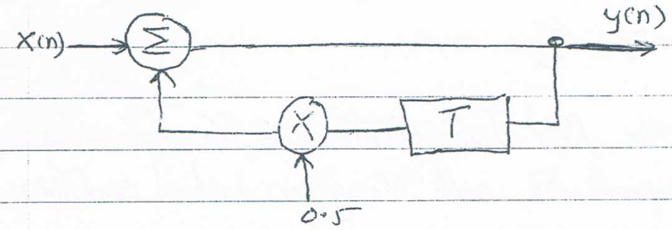
$$x[k+2] = \frac{1}{2}\left[x[k+1] - x[k] - k^2\right]$$

$n=0 \quad x[2] = \frac{1}{2}[x[1] - x[0] - 0] = \frac{1}{2}[3-1] = 1$

$n=1 \quad x[3] = \frac{1}{2}[x[2] - x[1] - 1] = \frac{1}{2}[1-3-1] = -\frac{3}{2}$

$n=2 \quad x[4] = \frac{1}{2}[x[3] - x[2] - 4] = \frac{1}{2}[-\frac{3}{2} - 1 - 4] = -\frac{13}{4}$

EX: For the following block diagram



1) Derive the difference equation
2) If $x(n) = u(n)$, calculate the output $y(n)$
   if $y(n) = 0$ for $n \leq -1$.
3) sketch output $y(n)$

Solution

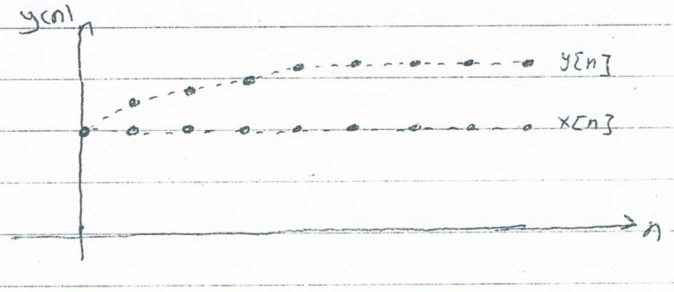
① $y(n) = x(n) + 0.5\, y[n-1]$

② $y(n) = u(n) + 0.5\, y[n-1]$

$y(0) = u(0) + 0.5\, y(-1) = 1 + 0.5 \times 0 = 1$

$y(1) = u(1) + 0.5\, y(0) = 1 + 0.5 \times 1 = 1.5$

n	0	1	2	3	4	5	6	7	8	9	10
y(n)	1	1.5	1.75	1.88	1.94	1.97	1.99	2.0	2.0	2	2



Home Work

H.W

1- If $z[n] \cdot z[n-1] = n^2 \quad z[1] = 7$

find $z[2]$, $z[3]$ and $z[4]$

ans. $\frac{4}{7}, \frac{63}{4}, \frac{64}{63}$

2. Calculate the first five terms of the following difference equations with the given initial conditions

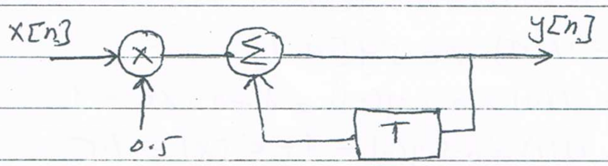
a) $x[n+1] - x[n] = 2, \quad x[0] = 3$

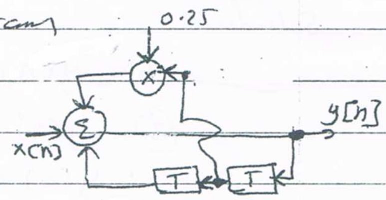
b) $x[n+2] + x[n+1] - x[n] = 4, \quad x[0] = 5, x[1] = 7$

ans. a) 3, 5, 7, 9, 11

b) 5, 7, 2, 9, -3

3 For the following blocks diagram





1- Derive the difference equations

2- If the input sequence [x[n]] equal to delta sequence ($\delta[n]$), if $y(n) = 0$ for $n < 1$. calculate the first four of the output $y[n]$.

Z-transform for Discrete system

The Z-transform methods apply when the variables being measured are discrete, if the signal is X[k], so the Z-transform of X[k] is:

$$\boxed{X(z) = \sum_{k=-\infty}^{\infty} x[k] \cdot z^{-k}} \leftarrow \text{two sided transform.}$$

and for positive side Z-transform

$$\boxed{X(z) = \sum_{k=0}^{\infty} x[k] \cdot z^{-k}}$$

finally for negative side Z-transform

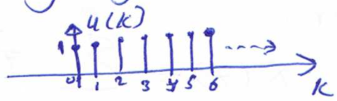
$$\boxed{X(z) = \sum_{k=-\infty}^{0} x[k] \cdot z^{-k}}$$

Rule: for the geometric series

$$\sum_{k=n_1}^{k=n_2} (z^{-1})^k = \frac{(z^{-1})^{n_1} - (z^{-1})^{n_2+1}}{1 - z^{-1}}$$

Ex: Find the Z-transform for the following signals

1- $x[k] = u[k]$



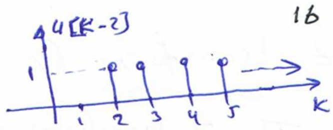
$$X[z] = \sum_{k=0}^{\infty} u[k]\, z^{-k} = u(0) + z^{-1} + z^{-2} + z^{-3} + \dots$$

$$= \sum_{k=0}^{\infty} (z^{-1})^k = \frac{(z^{-1})^0 - (z^{-1})^{\infty}}{1 - z^{-1}} = \frac{1-0}{1-z^{-1}} = \frac{1}{1-z^{-1}}$$

$$= \frac{z}{z-1}$$

Region of convergence (Roc): $|z^{-1}| < 1 \Rightarrow \frac{1}{|z|} < 1 \Rightarrow |z| > 1$

2. $X[k] = u[k-2]$

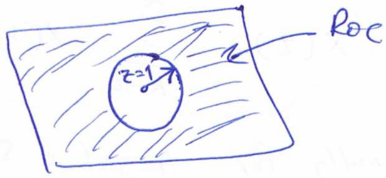


$$X(z) = \sum_{k=2}^{\infty} u[k-2]\, z^{-k} = \sum_{k=2}^{\infty} 1\,(z^{-1})^{k}$$

$$= \frac{(z^{-1})^{2} - (z^{-1})^{\infty}}{1-z^{-1}} = \frac{z^{-2} - 0}{1-z^{-1}} = \frac{z^{-2}}{1-z^{-1}} = \frac{1}{z^{2}(1-z^{-1})} = \frac{1}{z(z-1)}$$

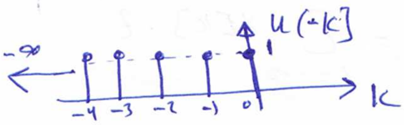
or $X(z) = -1 - z^{-1} + \sum_{k=0}^{\infty} z^{-k} = -1 - z^{-1} + \sum_{k=0}^{\infty} (z^{-1})^{k} = -1 + z^{-1} + \frac{1}{1-z^{-1}}$

RoC: $|z^{-1}| < 1 \implies |z| > 1$



3 - $X[k] = u[-k]$

$$X(z) = \sum_{k=-\infty}^{0} u[-k]\, z^{-k}$$



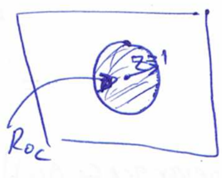
$$X(z) = \sum_{k=-\infty}^{0} 1\cdot (z^{-1})^{k} = \frac{(z^{-1})^{-\infty} - (z^{-1})^{0+1}}{1-z^{-1}} = \frac{-z^{-1}}{1-z^{-1}} = \frac{z^{-1}}{z^{-1}-1}$$

$$= \frac{1}{z(z^{-1}-1)} = \frac{1}{1-z}$$

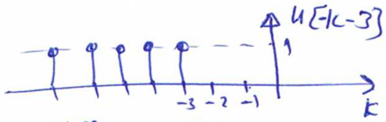
or

$$X(z) = \sum_{k=-\infty}^{0} z^{-k} \Big|_{k=-k} = \sum_{k=0}^{\infty} z^{k} = \frac{(z)^{0} - (z)^{\infty}}{1-z} = \frac{1}{1-z}$$

RoC: $|z| < 1$



4- $x[k] = u[-k-3]$



$$X(z) = \sum_{k=-\infty}^{-3} z^{-k} = \sum_{k=-\infty}^{-3} (z^{-1})^{k} = \frac{(z^{-1})^{-\infty} - (z^{-1})^{-3+1}}{1-z^{-1}} = \frac{0 - z^{2}}{1-z^{-1}}$$

or

$$X(z) = \sum_{k=-\infty}^{-3} z^{-k} \Big|_{k=-k} = \sum_{k=3}^{\infty} z^{k} = \frac{z^{3} - z^{\infty}}{1-z} = \frac{z^{3}}{1-z} = \frac{z^{3}}{z(z^{-1}-1)}$$

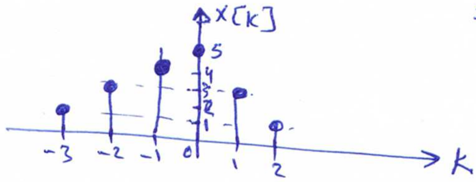
$$= \frac{z^{2}}{(z^{-1}-1)} = \frac{z^{2}}{-1(1-z^{-1})} = \frac{-z^{2}}{1-z^{-1}}$$

or

$$X(z) = \sum_{k=-\infty}^{-3} z^{-k} \Big|_{k=-k} = \sum_{k=3}^{\infty} z^{k} = -1 - z - z^{2} + \sum_{k=0}^{\infty} z^{k}$$

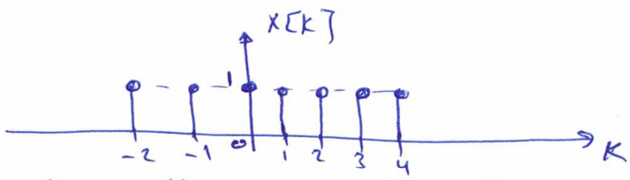
$$= -1 - z - z^{2} + \frac{1}{1-z}$$

5-



$$X(z) = \sum_{k=-3}^{2} x[k] \cdot z^{-k} = z^{3} + 3z^{2} + 4z + 5z^{0} + 3z^{-1} + z^{-2}$$

6-



$$X(z) = \sum_{k=-2}^{4} x[k] \cdot z^{-k} = \sum_{k=-2}^{4} (z^{-1})^{k} = \frac{(z^{-1})^{-2} - (z^{-1})^{4+1}}{1-z^{-1}}$$

$$= \frac{z^{2} - z^{-5}}{1-z^{-1}}$$

7- $f[k] = \cos(wk) \cdot u(k)$

$$\cos[wk] = \frac{e^{jwk} + e^{-jwk}}{2} = \frac{1}{2}\left(e^{jwk} + e^{-jwk}\right)$$

$$F(z) = \mathcal{Z}\left[f[k]\right] = \sum_{k=-\infty}^{\infty} f[k] \cdot z^{-k} = \sum_{k=-\infty}^{\infty} \cos[wk] \cdot u[k] \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} \cos[wk] \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{1}{2}\left(e^{jwk} + e^{-jwk}\right) \cdot z^{-k}$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} e^{jwk} \cdot z^{-k} + \sum_{k=0}^{\infty} e^{-jwk} \cdot z^{-k}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty}\left(e^{jw} \cdot z^{-1}\right)^{k} + \sum_{k=0}^{\infty}\left(e^{-jw} \cdot z^{-1}\right)^{k}\right] = \frac{1}{2}\left[\frac{1}{1-e^{jw}z^{-1}} + \frac{1}{1-e^{-jw}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{jw}} + \frac{z}{z-e^{-jw}}\right] = \frac{1}{2}\left[\frac{z(z-e^{-jw}) + z(z-e^{jw})}{(z-e^{jw})(z-e^{-jw})}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - z \cdot e^{-jw} + z^2 - z \cdot e^{jw}}{(z-e^{jw})(z-e^{-jw})}\right] = \frac{1}{2}\left[\frac{2z^2 - z(e^{jw} + e^{-jw})}{z^2 - z \cdot e^{-jw} - z \cdot e^{jw} + e^{jw} \cdot e^{-jw}}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{jw} + e^{-jw})}{z^2 - z(e^{jw} + e^{-jw}) + 1}\right]$$

$$\boxed{\text{But } \cos w = \frac{e^{jw} + e^{-jw}}{2} \quad \therefore e^{jw} + e^{-jw} = 2\cos w}$$

$$= \frac{1}{2}\left[\frac{2z^2 - z \cdot 2 \cdot \cos w}{z^2 - z \cdot 2\cos w + 1}\right]$$

$$F(z) = \frac{1}{2}\left[\frac{2(z^2 - z \cdot \cos w)}{z^2 - 2z\cos w + 1}\right] = \frac{z^2 - z \cdot \cos w}{z^2 - 2z\cos w + 1}$$

if $w = \pi \Rightarrow \cos w = \cos\pi = -1$

$$\therefore F(z) = \frac{z^2 + z}{z^2 + 2z + 1}$$

H.W: Find the Z-transform of

1- $f[k] = \sin(wk) \cdot u[k]$

2. $f[k] = e^{-k} \cdot u[k]$

Rule: If $f[k] = k^n \cdot d[k]$

$$\therefore \mathcal{Z}\left[f[k]\right] = \left(-z\frac{d}{dz}\right)^n \cdot D(z)$$

where $D(z) = \mathcal{Z}\left[d[k]\right] = \sum_{k=-\infty}^{\infty} d[k] \cdot z^{-k}$

Ex: Find the Z-transform of the following sequences

1- $f[k] = K \cdot u[k]$

$d[k] = u[k]$

$$\therefore D[z] = \mathcal{Z}\left[d(k)\right] = \sum_{k=-\infty}^{\infty} d(k) \cdot z^{-k} = \sum_{k=0}^{\infty} u[k]\, z^{-k} = \frac{1}{1-z^{-1}}$$

$$\therefore D(z) = \frac{z}{z-1}$$

$n = 1$

$$\mathcal{Z}\left[f[k]\right] = \left(-z\frac{d}{dz}\right)^1 \cdot D(z) = -z\frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$\therefore F(z) = -z\left[\frac{(z-1)-z\cdot 1}{(z-1)^2}\right] = -z\left[\frac{-1}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$\therefore F(z) = \frac{z}{(z-1)^2}$$

2. $f[k] = k^2 \cdot u[k]$

$$d[k] = u[k] \Longrightarrow D(z) = \mathcal{Z}\left[u(k)\right] = \frac{z}{z-1}$$

$\boxed{n=2}$

$$\therefore F(z) = \left(-z\frac{d}{dz}\right)^n \cdot D(z) = \left(-z\frac{d}{dz}\right)^2 \cdot \frac{z}{z-1}$$

$$F(z) = -z\cdot\frac{d}{dz}\left[-z\cdot\frac{d}{dz}\left[\frac{z}{z-1}\right]\right]$$

$$= -z\cdot\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = -z\left[\frac{(z-1)^2 - z.2(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{\cancel{(z-1)}\left[(z-1)-2z\right]}{(z-1)^{\cancel{4}3}}\right] = -z\left[\frac{z-1-2z}{(z-1)^3}\right] = -z\left[\frac{(-1-z)}{(z-1)^3}\right]$$

$$\therefore F(z) = \frac{z(z+1)}{(z-1)^3}$$

3- $f[k] = k^2 \cdot a^k \cdot u[k]$

$$d[k] = a^k \cdot u[k] \Longrightarrow D[z] = \mathcal{Z}\left[d[k]\right] = \sum_{k=-\infty}^{\infty} u[k]\cdot a^k \cdot z^{-k}$$

$$\therefore D(z) = \sum_{k=0}^{\infty} a^k \cdot z^{-k} = \sum_{k=0}^{\infty} (a\cdot z^{-1})^k = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$F(z) = \left(-z\frac{d}{dz}\right)^n \cdot D(z)$$

$n=2$

$$F(z) = \left(-z\frac{d}{dz}\right)^2 \cdot D(z) = -z\cdot\frac{d}{dz}\left[-z\frac{d}{dz}\left[D(z)\right]\right]$$

$$= -z\frac{d}{dz}\left[-z\frac{d}{dz}\left(\frac{z}{z-a}\right)\right] = -z\frac{d}{dz}\left[-z.\frac{(z-a)-z}{(z-a)^2}\right]$$

$$= -z\frac{d}{dz}\left[\frac{az}{(z-a)^2}\right] = -z\left[\frac{a(z-a)^2 - 2az(z-a)}{(z-a)^4}\right]$$

$$F(z) = -z\left[\frac{\cancel{(z-a)}\left[a(z-a)-2az\right]}{(z-a)^{\cancel{4}\,3}}\right] = -z\left[\frac{a(z-a)-2az}{(z-a)^3}\right]$$

$$F(z) = -z\left[\frac{az-a^2-2az}{(z-a)^3}\right] = -z\left[\frac{-a^2-az}{(z-a)^3}\right]$$

$$F(z) = \frac{az(z+a)}{(z-a)^3}$$

$$\therefore F(z) = \frac{az(z+a)}{(z-a)^3}$$

H.W: Find the Z-transform of the following sequence

$$f[k] = k^3 \cdot \sin[wk] \cdot a^{2k} \cdot u[k]$$

Properties of the Z-transform

The properties of the Z-transform are
1- linearity
2- Shift theorems
3- The complex translation theorem.

1- linearity

If $f[k]$ and $g[k]$ are two sequences then

$$\mathcal{Z}\left[f[k]+g[k]\right] = \mathcal{Z}\left[f[k]\right] + \mathcal{Z}\left[g[k]\right]$$

Ex: Find the Z-transform of $e^{-k}+3k$

Solution $\mathcal{Z}[e^{-k}+3k] = \mathcal{Z}[e^{-k}] + 3\mathcal{Z}[k]$

$$\mathcal{Z}[e^{-k}] = \frac{z}{z-e^{-1}}$$

$$\mathcal{Z}[k] = \frac{z}{(z-1)^2}$$

$$\mathcal{Z}[e^{-k}+3k] = \frac{z}{z-e^{-1}} + \frac{3z}{(z-1)^2}$$

2- First shift theorem

If $F(z) = \mathcal{Z}[f[k]]$, then

$$\mathcal{Z}\left[f[k+i]\right] = z^i F(z) - \left(z^i . f[0] + z^{i-1} f[1] + \cdots + z f[i-1]\right)$$

if $i=1$

$$\mathcal{Z}\left[f[k+1]\right] = z F(z) - z . f[0]$$

if $i=2$ $\quad \mathcal{Z}\left[f[k+2]\right] = z^2 F(z) - z^2 f[0] - z f[1]$

<u>Example</u>: The sequence $f[k]$ is defined by

$$f[k] = \begin{cases} 0 & k = 0,1,2,3 \\ 1 & k = 4,5,6,\ldots \end{cases}$$

Write down the sequence $f[k+1]$ and verify that

$$\mathcal{Z}\left[f[k+1]\right] = z\,F(z) - z\,f[0]$$

Where $F(z)$ is the z-transform of $f[k]$.

<u>Solution</u>

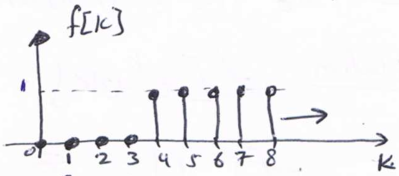
$$F(z) = \sum_{k=4}^{\infty} z^{-k} = \sum_{k=4}^{\infty} (z^{-1})^{k} = \frac{(z^{-1})^{4} - (z^{-1})^{\infty}}{1 - z^{-1}}$$

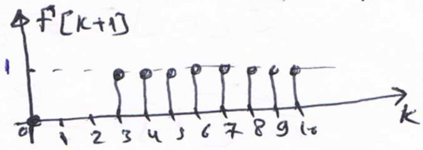
$$F(z) = \frac{z^{-4}}{1 - z^{-1}} = \frac{1}{z^{4}(1 - z^{-1})} = \frac{1}{z^{3}(z-1)}$$

$$\mathcal{Z}\left[f[k+1]\right] = \sum_{k=3}^{\infty} z^{-k} = \sum_{k=3}^{\infty} (z^{-1})^{k}$$

$$= \frac{(z^{-1})^{3} - (z^{-1})^{\infty}}{1 - z^{-1}} = \frac{z^{-3}}{1 - z^{-1}}$$

$$= \frac{1}{z^{3}(1 - z^{-1})} = \frac{1}{z^{2}(z-1)}$$





$$\mathcal{Z}\left[f[k+1]\right] = z\,F(z) - z\,f[0]$$

$$\frac{1}{z^{2}(z-1)} = \cancel{z} \cdot \frac{1}{z^{\cancel{3}\,2}(z-1)} - z \cdot 0$$

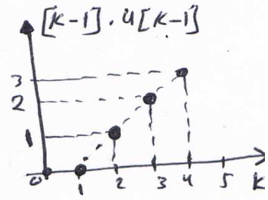
$$\frac{1}{z^{2}(z-1)} = \frac{1}{z^{2}(z-1)}$$

3. Second shift theorem

If $F(z) = \mathcal{Z}\left[f[k]\right]$

$$\mathcal{Z}\Big[f[k-i]\cdot u[k-i]\Big] = z^{-i}\cdot F(z)$$

$f[k] = k.u[k]$



Example: If $f[k] = k\cdot u[k]$, find the z-transform of $[k-1]\cdot u[k-1]$

Solution

$$F(z) = \mathcal{Z}\left[f[k]\right] = \mathcal{Z}\left[k\right] = \frac{z}{(z-1)^2}$$

$$\mathcal{Z}\left[(k-1)u[k-1]\right] = z^{-1}\cdot F(z) = z^{-1}\cdot\frac{z}{(z-1)^2}$$

$$= \frac{1}{(z-1)^2}$$

Example: Find the z-transform of $u[k]$ and shifted unit step $u[k-2]$

Solution

$$\mathcal{Z}\left[u[k]\right] = \frac{z}{z-1}$$

$$\mathcal{Z}\left[u[k-2]\right] = z^{-2}\,\mathcal{Z}\left[u[k]\right] = z^{-2}\cdot\frac{z}{z-1} = \frac{z^{-1}}{z-1}$$

$$= \frac{1}{z(z-1)}$$

Example: Find the sequence whose z-transform is

i) $\frac{1}{z-1}$ ii) $\frac{1}{z^2(z-1)^2}$

u[k]

<u>Solution</u>

i- $$\frac{1}{z-1} = \frac{1}{z-1}\cdot\frac{z}{z} = \frac{z}{z(z-1)} = \frac{1}{z}\cdot\frac{z}{(z-1)}$$

$$= z^{-1}\,\frac{z}{z-1}$$

so, $$\mathcal{Z}\left[u[k-1]\right] = z^{-1}\cdot\frac{z}{z-1}$$

$\therefore$ The required sequence is therefore $u[k-1]$.

[k

ii- $$\frac{1}{z^2(z-1)^2} = \frac{1}{z^2(z-1)^2}\cdot\frac{z}{z} = \frac{z}{z^3(z-1)^2} = \frac{1}{z^3}\cdot\frac{z}{(z-1)^2}$$

$$= z^{-3}\,\frac{z}{(z-1)^2}$$

$$\therefore\ \mathcal{Z}\left[u[k-3]\right] = z^{-3}\frac{z}{(z-1)^2}$$

[k-3]

From the second shift property, $z^{-3}\frac{z}{(z-1)^2}$ is the z-transform of $[k-3]\cdot u[k-3]$.

4- The complex translation theorem

If $F(z)$ is the z-transform of $f[k]$

$$\therefore \quad \boxed{\mathcal{Z}\left[e^{-bk} f[k]\right] = F(z)\Big|_{z \Rightarrow e^{b} z} = F(e^{b} z)}$$

Ex: Given that the z-transform of $\cos[ak]$ is

$$F(z) = \frac{z(z-\cos a)}{z^2 - 2z\cos a + 1}$$

Find the z-transform of $e^{-2k} \cdot \cos(ak)$

Solution $\quad b=2$ and $z \to e^{2} z$

$$F(e^{2}z) = \frac{e^{2}z(e^{2}z - \cos a)}{e^{4}z^{2} - 2e^{2}z\cos a + 1}$$

H.W1: use the complex translation theorem to find the z-transform of a) $e^{-bk} \cdot k$ b) $e^{-k} \sin[k]$

ans: a- $\dfrac{z e^{b}}{(z e^{b} - 1)^2}$ b- $\dfrac{e z \sin(1)}{e^{2} z^{2} - 2ez\cos[1] + 1}$

H.W2: If $f[k] = 4(3)^k$, find $\mathcal{Z}[f[k]]$. use the first shift theorem to deduce $\mathcal{Z}[f[k+1]]$. show that

$$\mathcal{Z}[f[k+1]] - 3\mathcal{Z}[f[k]] = 0.$$

ans: $\mathcal{Z}[4(3)^k] = 4\dfrac{z}{z-3}$, $\mathcal{Z}[f[k+1]] = 12 \cdot \dfrac{z}{z-3}$

Exercises of Z-transform

1- Using the definition of the Z-transform to prove the followi[ng] table

$f(k)$	$F(z)$
1- $\delta(k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$	1
2- $u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$	$\frac{z}{z-1}$
3- $u(-k) = \begin{cases} 1 & k \leq 0 \\ 0 & k > 0 \end{cases}$	$\frac{1}{1-z}$
4- $u(-k-2)$	$\frac{z^2}{1-z}$
5- $k\, u(k)$	$\frac{z}{(z-1)^2}$
6- $e^{-ak}\, u(k)$	$\frac{z}{z-e^{-a}}$
7- $a^k\, u(k)$	$\frac{z}{z-a}$
8- $k \cdot a^k\, u(k)$	$\frac{az}{(z-a)^2}$
9- $k^2 a^k\, u(k)$	$\frac{az(z+a)}{(z-a)^3}$
10- $k^3\, u(k)$	$\frac{z(z^2+4z+1)}{(z-1)^4}$

11 - $\sin ak \; u(k)$ $\qquad \dfrac{z \sin a}{z^2 - 2z \cos a + 1}$

12 - $\cos ak \; u(k)$ $\qquad \dfrac{z(z - \cos a)}{z^2 - 2z \cos a + 1}$

13 - $\sin bk \cdot e^{-ak} \cdot u(k)$ $\qquad \dfrac{z e^{-a} \sin b}{z^2 - 2z e^{-a} \cos b + e^{-2a}}$

14 - $\cos bk \cdot e^{-ak} \cdot u(k)$ $\qquad \dfrac{z^2 - z e^{-a} \cos b}{z^2 - 2z e^{-a} \cos b + e^{-2a}}$

15 - $f(k) = \begin{cases} 1 & -2 \leq k \leq 4 \\ 0 & \text{elsewhere} \end{cases}$ $\qquad \dfrac{z^3 - z^{-3}}{z - 1}$

2. Using the definition of the z-transform, find closed-form expression for the z-transforms of the following sequences $f(k)$ where

a- $f(0) = 0$, $f(1) = 0$, $f(k) = 1$ for $k \geq 2$

b- $f(k) = 3k$, $k \geq 0$

c- $f(k) = \begin{cases} 0 & k = 0, 1, \dots, 5 \\ 4 & k > 5 \end{cases}$

d- $f(k) = e^{-k}$, $k = 0, 1, 2, \dots$

e- $f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(k) = 0$ for $k \geq 3$

f- $f(0) = 3$, $f(k) = 0$ for $k \neq 0$

3 - Find the Z-transform of the following sequences

a - $\cos 3k \cdot u(k)$ b - $e^{k}\, u(k)$ c - $e^{-2k} \cos k\, u(k)$ d - $e^{4k} \sin 2k\, u(k)$ e - $4^{k}\, u(k)$

f - $(-3)^{k}\, u(k)$ g - $\sin(\frac{\pi}{2} k)\, u(k)$ h - $\cos(\frac{\pi}{2} k)\, u(k)$

4 - Find the Z-transform of

(a) $3(4)^{k} + 7k^{2}, \quad k \geq 0$

(b) $3e^{-k} \cdot \sin 4k - k, \quad k \geq 0$

(c) $u(k-4)$

(d) $(k-3) \cdot u(k-3)$

5 - prove that the Z-transform of $e^{-at} f(t)\, u(t)$ is $F(e^{aT} z)$.

6 - use the complex translation theorem to find the Z-transform of

a - $k \cdot e^{-bk}\, u(k)$ b - $e^{-k} \sin k\, u(k)$

7 - If $f(k) = (3)^{k} \cdot 4$, find $\mathcal{Z}[f[k]]$. use the Last formula to deduce $\mathcal{Z}[f[k+1]]$. show that $\mathcal{Z}[f[k+1]] - 3\mathcal{Z}[f[k]] =$

8 - Write down the first five terms of the sequence defined by $f[k] = 4(2)^{k-1} \cdot u[k-1]$, for $k \geq 0$.
Find its Z-transform.

Solutions

2- a- $\dfrac{1}{z(z-1)}$ b- $\dfrac{3z}{(z-1)^2}$ c- $\dfrac{4}{z^5(z-1)}$ d- $\dfrac{ez}{ez-1}$

e- $\dfrac{z^2+2z+3}{z^2}$ f- 3

3- a- $\dfrac{z(z-\cos 3)}{z^2-2z\cos 3+1}$ b- $\dfrac{z}{z-e}$ c- $\dfrac{z^2-ze^{-2}\cos 1}{z^2-2ze^{-2}\cos 1+e^{-4}}$

d- $\dfrac{ze^{-4}\sin 2}{z^2-2ze^{-4}\cos 2+e^{-8}}$ e- $\dfrac{z}{z-4}$ f- $\dfrac{z}{z+3}$

g- $\dfrac{z}{z^2+1}$ h- $\dfrac{z^2}{z^2+1}$

4- a- $\dfrac{3z}{z-4}+\dfrac{7z(z+1)}{(z-1)^3}$ b- $\dfrac{3ze^{-1}\sin 4}{z^2-2ze^{-1}\cos 4+e^{-2}}-\dfrac{z}{(z-1)^2}$

c- $\dfrac{1}{z^3(z-1)}$ d- $\dfrac{1}{z^2(z-1)^2}$

6- a- $\dfrac{ze^{b}}{(e^{b}z-1)^2}$ b- $\dfrac{e^{1}z\sin 1}{e^{2}z^2-2ez\cos 1+1}$

7- $\dfrac{4z}{z-3}$, $\dfrac{12z}{z-3}$

8- 0, 4, 8, 16, 32. $\dfrac{4}{z-2}$

Inversion of Z-transform

We can make the inversion of Z-transform use of tables of transforms, partial fractions and the shift theorems.

Ex: Find the inverse Z-transform of the following functions..

1- $F(z) = \dfrac{z}{z-4} = \dfrac{1}{1-4z^{-1}}$

$\therefore f[k] = \mathcal{Z}^{-1}[F(z)] = 4^{k} \cdot u[k]$

2- $F(z) = \dfrac{z}{z-1}$

$F(z) = \dfrac{1}{1-z^{-1}}$

$\therefore F[k] = \mathcal{Z}^{-1}[F(z)] = u[k]$

3- $F(z) = z + 2z^{-1} - 9z^{-2}$

$f[k] = \mathcal{Z}^{-1}[F(z)] = \mathcal{Z}^{-1}[z] + \mathcal{Z}^{-1}[2z^{-1}] - \mathcal{Z}^{-1}[9z^{-2}]$

$= \delta[k+1] + 2\,\delta[k-1] - 9\,\delta[k-2]$

4- $F(z) = \dfrac{z+3}{z-2}$

$F(z) = \dfrac{z}{z-2} + \dfrac{3}{z-2} = \dfrac{z}{z-2} + \dfrac{3z}{z(z-2)}$

$F(z) = \dfrac{z}{z-2} + 3z^{-1} \cdot \dfrac{z}{z-2}$

$f[k] = \mathcal{Z}^{-1}[F(z)] = \mathcal{Z}^{-1}\left[\dfrac{z}{z-2}\right] + \mathcal{Z}^{-1}\left[3z^{-1} \cdot \dfrac{z}{z-2}\right]$

$f[k] = 2^{k} u[k] + 3\, 2^{k-1} \cdot u[k-1]$

5- $F(z) = \dfrac{2z^2 - z}{(z-5)(z+4)} = \dfrac{z[2z-1]}{(z-5)(z+4)}$

$\therefore \dfrac{F(z)}{z} = \dfrac{2z-1}{(z-5)(z+4)} = \dfrac{A}{(z-5)} + \dfrac{B}{(z+4)}$

$\therefore A = \dfrac{F(z)}{z} \cdot (z-5)\Big|_{z=5} = \dfrac{2z-1}{z+4}\Big|_{z=5} = 1$

$B = \dfrac{F(z)}{z} \cdot (z+4)\Big|_{z=-4} = \dfrac{2z-1}{(z-5)}\Big|_{z=-4} = 1$

$\therefore \dfrac{F(z)}{z} = \dfrac{1}{(z-5)} + \dfrac{1}{(z+4)}$

$\therefore F(z) = \dfrac{z}{z-5} + \dfrac{z}{z+4} = \dfrac{1}{1-5z^{-1}} + \dfrac{1}{1+4z^{-1}}$

$\therefore f[k] = \mathcal{Z}^{-1}[F(z)] = \mathcal{Z}^{-1}\left[\dfrac{1}{1-5z^{-1}}\right] + \mathcal{Z}^{-1}\left[\dfrac{1}{1+4z^{-1}}\right]$

$f[k] = 5^k + (-4)^k$

6- $F(z) = \dfrac{1}{(z-1)(z+2)} \longrightarrow F(z) \cdot \dfrac{1}{(z-1)(z+2)} \cdot \dfrac{z}{z}$

$F(z) = \dfrac{z}{z(z-1)(z+2)}$

$\therefore \dfrac{F(z)}{z} = \dfrac{1}{z(z-1)(z+2)} = \dfrac{A}{z} + \dfrac{B}{(z-1)} + \dfrac{C}{z+2}$

$\therefore A = \dfrac{F(z)}{z} \cdot z\Big|_{z=0} = \dfrac{1}{(z-1)(z+2)}\Big|_{z=0} = -\dfrac{1}{2}$

$$B = \frac{F(z)}{z}\cdot(z-1)\Big|_{z=1} = \frac{1}{z(z+2)} = \frac{1}{3}$$

$$C = \frac{F(z)}{z}\cdot(z+2)\Big|_{z=-2} = \frac{1}{z(z-1)}\Big|_{z=-2} = \frac{1}{6}$$

$$\therefore \frac{F(z)}{z} = \frac{-\frac{1}{2}}{z} + \frac{\frac{1}{3}}{z-1} + \frac{1/6}{z+2}$$

$$\therefore F(z) = -\frac{1}{2} + \frac{1}{3}\,\frac{z}{z-1} + \frac{1}{6}\,\frac{z}{z+2}$$

$$f[k] = \mathcal{Z}^{-1}[F(z)] = -\frac{1}{2}\,\delta[k] + \frac{1}{3}\,u[k] + \frac{1}{6}\,(-2)^{k}\,u[k]$$

7- $F(z) = \dfrac{z}{1-5z^{-1}+6z^{-2}} \Rightarrow F(z) = \dfrac{z}{1-5z^{-1}+6z^{-2}}\,\dfrac{z^2}{z^2}$

$$F(z) = \frac{z^3}{z^2-5z+6}$$

$$\begin{array}{r|l} & z \\ z^2-5z+6 & z^3 \\ & z^3-5z^2+6z \\ \hline & 0+5z^2-6z \end{array}$$

$$F(z) = z + \frac{5z^2-6z}{z^2-5z+6}$$

$$F(z) = z + F_1(z)$$

$$\therefore F_1(z) = \frac{5z^2-6z}{z^2-5z+6} = \frac{z[5z-6]}{(z-3)(z-2)}$$

$$\frac{F_1(z)}{z} = \frac{5z-6}{(z-3)(z-2)} = \frac{A}{(z-3)} + \frac{B}{(z-2)}$$

$$A = \frac{F_1(z)}{z}\cdot(z-3)\Big|_{z=3} = 9$$

$$B = \frac{F_1(z)}{z}\cdot(z-2)\Big|_{z=2} = -4$$

$$\frac{F_1(z)}{z} = \frac{9}{z-3} - \frac{4}{z-2}$$

$$F_1(z) = 9\,\frac{z}{z-3} - 4\,\frac{z}{z-2}$$

$$F(z) = z + F_1(z) = z + 9\,\frac{z}{z-3} - 4\,\frac{z}{z-2}$$

$$f[k] = \mathcal{Z}^{-1}\left[F[z]\right] = \delta[k+1] + 9\,3^{k}u[k] - 4\,2^{k}u[k]$$

8. $$F(z) = \frac{z^3-2}{z^2+4z+4}$$

$$\begin{array}{r} z-4 \\ z^2+4z+4\,\overline{)\,z^3-2} \\ z^3+4z^2+4z \\ \hline -4z^2-4z-2 \\ -4z^2-16z-16 \\ \hline 12z+14 \end{array}$$

$$F(z) = z-4+\frac{12z+14}{z^2+4z+4}$$

$$F(z) = z-4+F_1(z)$$

$$F_1(z) = \frac{12z+14}{z^2+4z+4}\cdot\frac{z}{z}$$

$$F_1(z) = \frac{z(12z+14)}{z(z+2)^2}$$

$$\frac{F_1(z)}{z} = \frac{12z+14}{z(z+2)^2} = \frac{A}{z} + \frac{k_0}{(z+2)^2} + \frac{k_1}{z+2}$$

$$A = \frac{F_1(z)}{z}\cdot z\Big|_{z=0} = \frac{12z+14}{(z+2)^2}\Big|_{z=0} = \frac{14}{4} = \frac{7}{2}$$

$$k_0 = \frac{F_1(z)}{z}\cdot(z+2)^2\Big|_{z=-2} = \frac{12z+14}{z}\Big|_{z=-2} = \frac{-10}{-2} = +5$$

$$k_1 = \frac{d}{dz}\left[\frac{F_1(z)}{z}\cdot(z+2)^2\right]\Big|_{z=-2} = \frac{d}{dz}\left[\frac{12z+14}{z}\right]\Big|_{z=-2} = \frac{12z-(12z+14)\cdot}{z^2}\Big|_{z=-2} = \frac{-7}{2}$$

$$\frac{F_1(z)}{z} = \frac{7/2}{z} + 5\,\frac{1}{(z+2)^2} - \frac{7}{2}\;\frac{1}{z+2}$$

$$F_1(z) = \frac{7}{2} + 5\,\frac{z}{(z+2)^2} - \frac{7}{2}\;\frac{z}{z+2}$$

$$F(z) = z - 4 + F_1(z) = z \underline{-4 + \frac{7}{2}} + 5\,\frac{z}{(z+2)^2} - \frac{7}{2}\;\frac{z}{z+2}$$

$$F(z) = -\frac{1}{8} + z + 5\,\frac{z}{(z+2)^2} - \frac{7}{2}\;\frac{z}{z+2}$$

$$f[k] = \mathcal{Z}^{-1}[F(z)]$$

$$\boxed{\begin{aligned} f(k) &= k \cdot a^k u[k] \\ F(z) &= \frac{az}{(z-a)^2} \end{aligned}}$$

$$= -\frac{1}{8}\,\delta[k] + \delta[k+1] - \frac{5}{2}(-2)^k \cdot k\,u[k] - \frac{7}{2}\,(-2)^k u[k]$$

$$f[k] = -\frac{1}{8}\,\delta[k] + \delta[k+1] + \left[-\frac{5}{2}(-2)^k \cdot k\,u[k] - \frac{7}{2}(-2)^k u[k]\right]$$

<u>Exercises</u>: Find the inverse z-transform of the following

a) $\dfrac{z^2 + 2z}{3z^2 - 4z - 7}$ b) $\dfrac{2z^3 + z}{(z-3)^2(z-1)}$ c) $\dfrac{z^2}{(z^2 - \frac{1}{9})}$

d) $\dfrac{z+1}{(z-3)z^2}$

<u>Solutions</u>

a- $\frac{13}{30}\left(\frac{7}{3}\right)^k - \frac{1}{10}(-1)^k$

b- $\frac{19}{6}k(3)^k + \frac{3}{4}u[k] + \frac{5}{4}(3)^k$

c- $\frac{1}{2}\left(\frac{1}{3}\right)^k + \frac{1}{2}\left(\frac{-1}{3}\right)^k$

d- $(3)^{k-2}u[k-2] + (3)^{k-3} \cdot u[k-3]$

The z-transform and difference equations

The Laplace transform can be used to solve the linear, constant coefficient, ordinary differential equation, similarly the z-transform has a role to play in the solution of difference equations.

Ex: Solve the difference equation $y[k+1]-3y[k]=0$, $y[0]=4$.

<u>Solution</u>: Taking the z-transform of both sides of the equation, we have

$$\mathcal{Z}\left[y[k+1]\right]-3\mathcal{Z}\left[y[k]\right]=\mathcal{Z}[0]$$

But

$$\mathcal{Z}\left[y[k+1]\right]=z.Y(z)-z.y[0]=zY(z)-4z$$

$$\mathcal{Z}\left[y[k]\right]=Y(z)$$

$$\mathcal{Z}[0]=0$$

$$zY(z)-4z-3Y(z)=0$$

$$Y(z)[z-3]=4z$$

$$\therefore Y(z)=\frac{4z}{z-3}$$

$$y[k]=\mathcal{Z}^{-1}Y(z)=\mathcal{Z}^{-1}\left[\frac{4z}{z-3}\right]=4\,3^{k}$$

$$\therefore y[k]=4.3^{k}$$

Ex: Solve the second-order difference equation

$$y[k+2]-5y[k+1]+6y[k]=0 \qquad y[0]=0\ ,\ y[1]=2$$

Solution:

$$\mathcal{Z}\left[y[k+2]\right]-5\,\mathcal{Z}\left[y[k+1]\right]+6\,\mathcal{Z}\left[y[k]\right]=\mathcal{Z}[0]$$

$$\mathcal{Z}\left[y[k+2]\right]=z^2Y(z)-z^2y[0]-zy[1]=z^2Y(z)-2z$$

$$\mathcal{Z}\left[y[k+1]\right]=zY(z)-zy[0]=zY(z)$$

$$\mathcal{Z}\left[y[k]\right]=Y(z)$$

$$\mathcal{Z}[0]=0$$

$$z^2.Y(z)-2z-5zY(z)+6Y(z)=0$$

$$Y(z)\left[z^2-5z+6\right]=2z$$

$$\therefore Y(z)=\frac{2z}{z^2-5z+6}=\frac{2z}{(z-3)(z-2)}$$

$$\frac{Y(z)}{z}=\frac{2}{(z-3)(z-2)}$$

$$\frac{Y(z)}{z}=\frac{A}{(z-3)}+\frac{B}{z-2}$$

$$A=\frac{Y(z)}{z}.(z-3)\Big|_{z=3}=\frac{2}{z-2}\Big|_{z=3}=2$$

$$B=\frac{Y(z)}{z}\cdot(z-2)\Big|_{z=2}=\frac{2}{(z-3)}\Big|_{z=2}=-2$$

$$\therefore \frac{Y(z)}{z}=\frac{2}{(z-3)}-\frac{2}{(z-2)}$$

$$Y(z) = \frac{2z}{(z-3)} - \frac{2z}{z-2}$$

$$y[k] = \mathcal{Z}^{-1}[Y(z)] = 2(3)^k - 2(2)^k$$

Ex: For the following difference equation:

$$y[k+2] - 5y[k+1] + 6y[k] = k$$

1- What is the order of equation?
2- Is it linear or non-linear, homogenous or inhomogenous?
3- Reduce the highest argument of the dependent variable to k.
4- Solve the difference equation, if the initial conditions are $y[0]=0$, $y[1]=0$

Solution: 1- order = 2

2- linear, inhomogenous

3- $k \rightarrow k-2$

$$y[k] - 5y[k-1] + 6y[k-2] = k-2$$

4- $\mathcal{Z}[y[k+2]] - 5\mathcal{Z}[y[k+1]] + 6\mathcal{Z}[y[k]] = \mathcal{Z}[k]$

$$\mathcal{Z}[y[k+2]] = z^2 . Y(z) - z^2 . y[0] - z y[1] = z^2 Y(z)$$

$$\mathcal{Z}[y[k+1]] = z . Y(z) - z . y[0] = z . Y(z)$$

$$\mathcal{Z}[y[k]] = Y(z)$$

$$\mathcal{Z}[k] = \frac{z}{(z-1)^2}$$

$$z^2 . Y(z) - 5z Y(z) + 6 Y(z) = \frac{z}{(z-1)^2}$$

$$Y(z)[z^2 - 5z + 6] = \frac{z}{(z-1)^2}$$

$$Y(z) = \frac{z}{(z^2-5z+6)(z-1)^2} = \frac{z}{(z-2)(z-3)(z-1)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-2)(z-3)(z-1)^2}$$

$$\frac{Y(z)}{z} = \frac{A}{(z-2)} + \frac{B}{z-3} + \frac{K_0}{(z-1)^2} + \frac{K_1}{(z-1)}$$

$$A = \frac{Y(z)}{z}\cdot(z-2)\Big|_{z=2} = \frac{1}{(z-3)(z-1)^2}\Big|_{z=2} = -1$$

$$B = \frac{Y(z)}{z}\cdot(z-3)\Big|_{z=3} = \frac{1}{(z-2)(z-1)^2}\Big|_{z=3} = \frac{1}{4}$$

$$K_0 = \frac{Y(z)}{z}\cdot(z-1)^2\Big|_{z=1} = \frac{1}{(z-3)(z-2)}\Big|_{z=1} = \frac{1}{2}$$

$$K_1 = \frac{d}{dz}\left[\frac{Y(z)}{z}\cdot(z-1)^2\right]\Big|_{z=1} = \frac{d}{dz}\left[\frac{1}{(z-2)(z-3)}\right]\Big|_{z=1} = \frac{-2z+5}{[(z-2)(z-3)]^2}\Big|_{z=1}$$

$$K_1 = \frac{3}{4}$$

$$\frac{Y(z)}{z} = \frac{-1}{z-2} + \frac{\frac{1}{4}}{z-3} + \frac{1/2}{(z-1)^2} + \frac{3/4}{(z-1)}$$

$$\Rightarrow Y(z) = -\frac{z}{z-2} + \frac{1}{4}\frac{z}{z-3} + \frac{1}{2}\frac{z}{(z-1)^2} + \frac{3}{4}\frac{z}{(z-1)}$$

$$y[k] = \mathcal{Z}^{-1}Y(z) = -2^k + \frac{1}{4}3^k + \frac{1}{2}k + \frac{3}{4}u[k]$$

<u>Exercises</u> Solve the following difference equations.

1- $x[k+1]-3x[k]=-6$, $x[0]=1$

2- $2x[k+1]-x[k]=2^k$, $x[0]=2$

3- $x[k+1]+x[k]=2k+1$, $x[0]=0$

4- $x[k+2]-3x[k+1]+2x[k]=\delta[k]$, $x[0]=x[1]=0$

5- $x[k+2]-7x[k+1]+12x[k]=k$, $x[0]=1$, $x[1]=1$

<u>Solutions</u>

1- $x[k]=3-2\cdot(3)^k$

2. $x[k]=\frac{1}{3}\cdot(2)^k+\frac{5}{3}(0.5)^k$

3 - $x[k]=k$

4- $x[k]=(2)^{k-1}\cdot u[k-1]-u[k-1]$

5- $x[k]=\frac{1}{6}k-\frac{31}{36}u[k]+\frac{11}{3}(3)^k-\frac{17}{9}(4)^k$

Transfer function of z-transform

For the following block

transfer function $= H(z) = \dfrac{Y(z)}{X(z)}$

where

$X(z)$: input in z-domain $\quad X(z) = \mathcal{Z}[x[k]]$

$Y(z)$: output " " $\quad Y(z) = \mathcal{Z}[y[k]]$

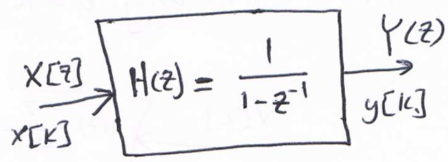
$H(z)$: transfer function in z-domain.

$h[k]$: impulse response $\quad h[k] = \mathcal{Z}^{-1}[H(z)]$

$x[k]$: input sequence $\quad x[k] = \mathcal{Z}^{-1}[X(z)]$

$y[k]$: output sequence $\quad y[k] = \mathcal{Z}^{-1}[Y(z)]$

Ex: for the following block, if $x[k] = 4 \cdot 2^k$ find $y[k]$.



Solution

$$X(z) = \mathcal{Z}[x[k]] = \frac{4z}{z-2}$$

$$H(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$\therefore Y(z) = H(z) \cdot X(z) = \left(\frac{z}{z-1}\right) \cdot \left(\frac{4z}{z-2}\right) = \frac{4z^2}{(z-1)(z-2)}$$

$$Y(z) = \frac{4z^2}{(z-1)(z-2)}$$

$$y[k] = \mathcal{Z}^{-1}[Y(z)$$

$$\frac{Y(z)}{z} = \frac{4z}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$A = \frac{Y(z)}{z}\cdot(z-1)\Big|_{z=1} = \frac{4z}{(z-2)}\Big|_{z=1} = -4$$

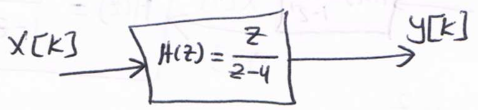
$$B = \frac{Y(z)}{z}\cdot(z-2)\Big|_{z=2} = \frac{4z}{(z-1)}\Big|_{z=2} = 8$$

$$\therefore \frac{Y(z)}{z} = \frac{-4}{z-1} + \frac{8}{z-2}$$

$$\therefore Y(z) = -4\,\frac{z}{z-1} + 8\,\frac{z}{z-2}$$

$$y[k] = \mathcal{Z}^{-1}\left[Y(z)\right] = -4\,u[k] + 8\,2^{k}\cdot u[k]$$

<u>H.W</u>: for the following block, if $x[k] = k\cdot 2^{k}$

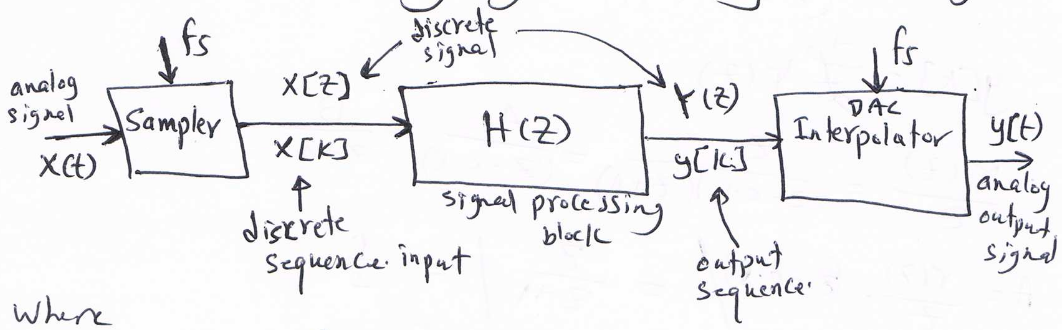


1- find $y[k]$.

2- Determine the impulse response.

System Application of Z-transform

Consider the following signal processing block diagram:



Where

f_s = sampling frequency , $f_s = \frac{1}{T_s} \Rightarrow T_s = \frac{1}{f_s}$

$$t = k \cdot T_s = k \cdot \frac{1}{f_s} = \frac{k}{f_s}$$

Ex: For the above block diagram, Find $y[k]$ and $y(t)$.
Assume that $f_s = 100Hz$, $f_0 = 50Hz$, $x(t) = \cos w_0 t$

$$H(z) = \frac{1}{1 - 0.5z^{-1}}$$

Solution

$$x(t) = \cos(2\pi f_0 t)$$

$$x[k] = \cos\left[2\pi f_0 \cdot \frac{k}{f_s}\right] = \cos\left[2\pi \times 50 \times \frac{k}{100}\right] = \cos[\pi k]$$

$$\therefore x[k] = \cos[\pi k]$$

$$X(z) = \mathcal{Z}\left[x[k]\right] = \frac{1 - z^{-1}\cos\pi}{1 - 2z^{-1}\cos\pi + z^{-2}} = \frac{1 + z^{-1}}{1 + 2z^{-1} + z^{-2}}$$

$$X(z) = \frac{1 + z^{-1}}{1 + 2z^{-1} + z^{-2}} \cdot \frac{z^2}{z^2} = \frac{z^2 + z}{z^2 + 2z + 1} = \frac{z\cancel{(z+1)}}{(z+1)^{\cancel{2}}}$$

$$X(z) = \frac{z}{z+1}$$

$$H(z) = \frac{Y(z)}{X(z)} \qquad H(z) = \frac{1}{1 - 0.5z^{-1}} \cdot \frac{z}{z} = \frac{z}{z - 0.5}$$

$$h[k] = \mathcal{Z}^{-1} H(z) = (0.5)^k$$

$$\therefore Y(z) = H(z) \cdot X(z) = \frac{z}{z-0.5} \cdot \frac{z}{z+1}$$

$$Y(z) = \frac{z^2}{(z-0.5)(z+1)}$$

$$y[k] = \mathcal{Z}^{-1} Y(z)$$

$$\frac{Y(z)}{z} = \frac{z}{(z-0.5)(z+1)} = \frac{A}{(z-0.5)} + \frac{B}{z+1}$$

$$A = \frac{Y(z)}{z} \cdot (z-0.5)\Big|_{z=0.5} = \frac{z}{z+1}\Big|_{z=0.5} = \frac{1}{3}$$

$$B = \frac{Y(z)}{z} \cdot (z+1)\Big|_{z=-1} = \frac{z}{z-0.5}\Big|_{z=1} = \frac{2}{3}$$

$$\frac{Y(z)}{z} = \frac{1/3}{z-0.5} + \frac{2/3}{z+1}$$

$$\therefore Y(z) = \frac{1}{3} \frac{z}{(z-0.5)} + \frac{2}{3} \frac{z}{z+1}$$

$$y[k] = \mathcal{Z}^{-1} Y[z] = \frac{1}{3}(0.5)^k + \frac{2}{3}(-1)^k$$

$$t = \frac{k}{f_s} \Rightarrow \boxed{k = f_s \cdot t = 100t}$$

$$y(t) = \frac{1}{3}(0.5)^{100t} + \frac{2}{3}(-1)^{100t}$$

$$h(t) = (0.5)^{100t}$$

__H.W__: If $x(t) = 50t$, $f_s = 75\ Hz$ and $H(z) = \frac{1}{z^2 - 5z + 6}$

find $h[k]$, $h(t)$, $y[k]$ and $y(t)$

END OF LECTURE

ANY QUESTIONS?